Newton's *Principia* : Proposition 1, Theorem 1

This essay gives a brief account of Proposition 1, Theorem 1 from Isaac Newton's *Philosophia naturalis principia mathematica* (1687)[1], a work universally referred to as the *Principia*. Following a few biographical details, we discuss the *Principia* generally, and after considering the passage itself, close with some brief remarks comparing this work to earlier works in the field.

Isaac Newton was born on Christmas Day 1642 in Woolsthorpe, Lincolnshire. As an undergraduate, Newton taught himself mathematics by reading the works of Euclid, Kepler and Descartes among others. He returned to Woolsthorpe soon after the completion of his degree in 1665 when the Cambridge colleges were closed following an outbreak of the plague. During the next twenty months, Newton was at his most prolific. In the *anni mirabilis* (miraculous years) of 1665-6, Newton "laid the foundations of all his major discoveries – on infinite series, the calculus, optics and gravitation."[2]

Circulating his work privately, Newton became known as a talented mathematician. He was appointed a fellow of Trinity College, Cambridge in 1667, and in 1669 succeeded Isaac Barrow to become the second Lucasian Professor of Mathematics. Newton later moved to London, occupying important positions at the Mint. Newton died in 1727, at the age of 84.

Newton's distaste for publication, along with the real difficulties in getting such material printed, meant that many of his books and tracts were published long after the important discoveries had been made. In this respect the *Principia* is somewhat unusual, being published soon after the work was completed.

In 1684 Edmond Halley (1656-1743) visited Newton in Cambridge, and asked what curve a planet would describe if attracted towards the Sun by a force inversely proportional to its

[1] The title may be rendered in English as *The Mathematical Principles of Natural Philosophy*.
[2] Stuart Hollingdale, *Makers of Mathematics* (London: Penguin, 1994), p. 171.

distance from the Sun. Newton replied that the curve would be an ellipse. He could not immediately supply his working, but soon sent Halley a short tract, *Du motu corporum*, containing the details. This was circulated among members of the Royal Society and caused something of a stir. Halley returned to Cambridge to urge Newton to expand and publish this tract. Newton had already embarked on the project, and hardly ceased work until the *Principia* was complete. However, the third part of the work was nearly withdrawn when one of several priority disputes broke out between Newton and Robert Hooke (1636-1703). Fortunately, Halley brought Newton round and saw the work to publication in 1687.

The *Principia* is arranged in three books, has an axiomatic structure, and is written in the Greek geometric style. The first book is preceded by a list of definitions and by Newton's three basic axioms – the three laws of motion – and some simple corollaries. The first of these corollaries, concerning the 'parallelogram of forces,' will be important later.[3] Along with an extended discussion of the 'method of first and last ratios'[4] which Newton put to use in the remainder of the work, the first book shows how to determine the shape of a planet's orbit given *any* law of centripetal force. In particular, he built on the earlier demonstration in *De motu*, showing that elliptical orbits and orbits in other conic sections imply, and are implied by, an inverse square law of attraction. Newton also demonstrated that planets moving in elliptical orbits under an inverse square law of force will obey Kepler's third law, according to which r^3/T^2 is a constant for all planets (where T is the time a planet takes to orbit the Sun, and r is the planet's mean distance from the Sun)[5]. The first book also included a demonstration that, under an inverse square law, a sphere of uniform density can be treated as a 'point mass', where the sphere's mass is concentrated at the centre of the sphere. This is important, for it entails that if a planet (or any other body) may be approximated by such a

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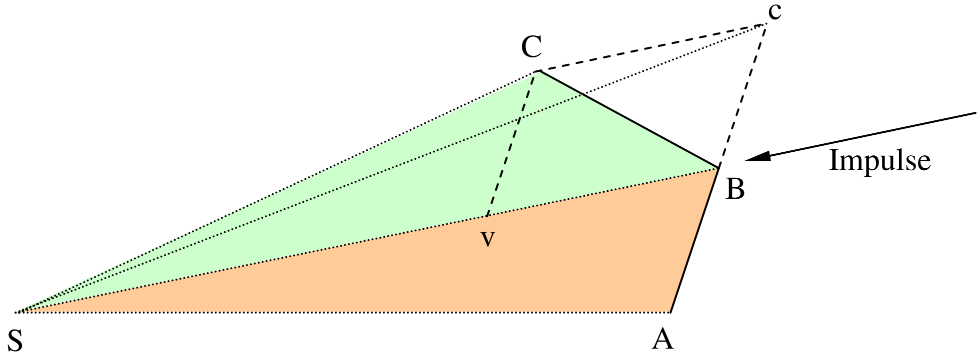
[3] See **SB** 12.B2 *Corollary I*, p. 390
[4] Parts of which appear at **SB** 12.B3, pp. 391-3 and **SB** 12.B4, pp. 393-4.
[5] The law is also known as the three-halves power law since it can be rearranged to give, $T = kr^{3/2}$, where k is constant across all planets.

sphere, it may also be approximated by a point mass. The second book is chiefly notable for its thorough refutation of Descartes' popular vortex theory of the solar system. In book three, Newton brings observational data to his previous results, investigating such diverse topics as comets, the moon's orbit, the tides, and the shape of the Earth.

The passage for discussion is reproduced at **SB** 12.B5, pp. 394-5 in a version based upon the first English translation of the *Principia* by Andrew Motte in 1729.[6] We saw above that Newton demonstrated connections between an inverse square law of attraction and Kepler's first and third laws. Proposition 1, Theorem 1 demonstrates the truth of Kepler's second law: as a planet orbits the Sun, the line joining the planet with the Sun will sweep out equal areas in equal times, or equivalently, the times taken for a planet to complete any section of its orbit are proportional to the areas such a line sweeps out in those times. Interestingly, Newton's demonstration holds for any possible centripetal law of force, and for orbits of all possible shapes.

To prove this result, Newton first simplifies the matter by approximating the continual force of attraction by a series of impulses directed towards the centre of the orbit (S) and acting on the planet at regular intervals. Suppose, then, that the planet begins at A and moves with constant velocity towards B, where it is struck by the first impulse, which diverts the planet towards C, where it is struck by the second impulse.



[6] Revised by F. Cajori in 1934.

The motion of the planet from B to C can be deduced from the ‘parallelogram of forces’. In the absence of the impulse, the planet’s momentum would have carried it to c in the same time it had taken to traverse AB; the impulse acting on its own would have moved the planet some distance in the direction of S, to v say. The result: the planet traverses the diagonal of the parallelogram BvCc, namely BC, in that same interval. Clearly the planet sweeps out the (coloured) areas ASB and BSC in equal intervals of time. But these two areas are equal. Newton shows this by first noting that triangles ASB and BSc have the same area (because the bases AB and Bc are the same length and the perpendicular distance to S is the same in both cases). And since the line SvB is parallel to Cc, from the rule that “triangles on the same base between the same parallels are equal in area,”[7] we know that the area of triangles BSC and BSc are equal. From these two equalities we deduce that the two coloured triangles are equal in area.

By reducing their breadth and increasing their number, the outer edge of these triangles may be made to ever more closely approximate the curved orbit of a planet, and the impulses ever more closely approximate the influence of a continual force. But since the above proof will hold for every step of this ever improving approximation, Newton, drawing on his earlier discussion of ‘first and last ratios,’ infers that in the limit “any described areas … [are] proportional to the times of description.”

As Jeremy Gray points out, Newton’s proof is only strictly valid for infinitesimal sections of a planet’s orbit. But the proof can be extended to cover finite portions of the planet’s orbit, and so the result is true as stated.[8]

Galileo Galilei’s (1564-1642) work on gravity extended only to the motion of bodies under constant acceleration, namely bodies in free-fall near the surface of the earth. Johannes

[7] MA 290 *TV and Audio – Cassette Notes*, p. 32.
[8] MA 290 Tape 6, Side 2: Jeremy Gray, “Newton’s *Principia* Theorem 1”

Kepler's (1571-1630) three laws were based primarily on observational data, and lacked further theoretical backing. Christiaan Huygens (1629-95) had investigated uniform circular motion but had not discussed motion in an ellipse. The *Principia* went far beyond these earlier writings. Although not immediately regarded as a classic scientific work, the publication of the *Principia* was a major event. It is now "universally acclaimed as one of the greatest scientific books ever written."[9]

[9] Stuart Hollingdale, *Makers of Mathematics*, p. 205.