Practical Problems and the Development of the Calculus between 1660 and 1760

1. Introduction

We begin by summarising the remit of what we now call the calculus and by briefly outlining the state of the 'pre-calculus' in 1660. After summarising the methods of the 'calculus proper' in the work of Newton and Leibniz, we then discuss the developments of these methods up to 1760. Drawing on this discussion, we finally assess the extent to which these developments were a response to 'practical problems'.

2. What is the Calculus?

For our purposes here, the calculus may be defined as the method or methods used in solving a certain range of problems. These problems include (i) finding instantaneous rates of change, (ii) finding tangents, (iii) locating maxima and minima, (iv) the rectification of curves, (v) finding the areas enclosed by curved lines and volumes by curved surfaces, and (vi) finding the centre of gravity of such areas and volumes. Problems of kinds (i) – (iii) are dealt with by what we now call the 'differential calculus,' and (iv) – (vi) by the 'integral calculus'.

3. The 'Pre-Calculus'

By 1660 there were several methods for attacking such problems. Perhaps the most notable were those of René Descartes (1596-1650) and Pierre de Fermat (1601-65). In his *La Géométrie* (1637), Descartes described a method for finding normals (and hence tangents) to a curve by finding the repeated roots of equations. Fermat's methods for finding tangents and maxima and minima were based on his ideas of quantities small enough to guarantee 'aedequality' (approximate equality) between two expressions. Making ingenius use of infinite geometric series, Fermat also found the areas under curves of the form $x^n y^m = constant$. There were many methods in use at this time, but none was sufficiently general in its application (or easy to use) so as to drive out the competition. Jeremy Gray's comment on Descartes' method is illustrative of the general situation: "this method was reliable in practice but it could be complicated to apply".[1]

4. Isaac Newton and Gottfried Leibniz

Isaac Newton (1642-1727) made his first discoveries in the calculus in 1665, refining his ideas in subsequent years. Gottfried Leibniz (1646-1716) made many of the same discoveries over the period 1672-6. Both used techniques for differentiation which differed little from those of Fermat. Leibniz' method of integration was more original and involved approximating the area under a curve by *summing* the areas of rectangles of equal width. This width was then reduced to zero, making the approximation exact. Newton and Leibniz improved on previous work by introducing *general* and *systematic* methods for solving area and tangent problems, and by revealing the inverse relationship between the processes used to solve these two types of problem. Perhaps the most striking difference between the work of Newton and Leibniz was Leibniz' introduction of a helpful notation, including the long script *s* for integration and *dy/dx* for the first derivative.

Newton published little and late. His earliest publication on the calculus, *De Analysi*, appeared in 1711. Leibniz published papers in *Acta Eruditorum* from 1684 onwards. These were often poorly written, and it fell largely to the Bernoulli family to clarify matters.

5. Uses and Developments of the Calculus

In this section we consider two ways in which the calculus developed: the study of differential equations, with special reference to the translation of Newton's *Principia* (1687) in the language of the calculus; and the origins of the variational calculus.

Several mathematicians were involved in translating the difficult geometrical arguments of Newton's *Principia* into the increasingly accessible language of the calculus. Among these

[1] MA 290 *TV5: Newton and Leibniz: The Birth of the Calculus*.

were Pierre Varignon (1654-1722), Leonhard Euler (1707-83) and Leibniz himself. Varignon used the calculus to derive laws of force from given orbits, and Euler to restate Newton's laws of motion and gravity in a new, more useful form. Leibniz applied the differential calculus to develop Newton's discussion of motion in resisting media, a topic with obvious practical applications to the motion of projectiles.[2] This last topic was later taken up by Benjamin Robins (1707-51) and Euler in their studies of gunnery.

Much of this 'translation' involved the formulation and solution of differential equations. Johann Bernoulli (1667-1748) developed techniques for translating geometric and mechanical problems into differential equations, which might then be solved (by integration) to yield a solution to the to the original problem. Johann had already used these methods to solve the catenary problem (of determining the shape assumed by a rope hanging between two fixed points), when his brother Jacob (1654-1705) set this challenge in 1691. An earlier problem of the same sort was posed by Claude Perrault (1631-85) in the 1670s: if you walk along a straight line, dragging an object not itself on that line, what curve is traced out by the object? The resulting curve, the tractrix, was studied by Newton, Leibniz, Christiaan Huygens (1629-95) and Johann Bernoulli.

During the 1730's Euler, and Johann and Daniel Bernoulli (1700-82) applied the same methods in their investigation of the vibrating clamped rod. After an initial disagreement with Alexis Clairaut (1713-65), Euler, in *Theoria motus lunaris* (1753), published his method for determining the orbit of the moon by approximating curves from a differential equation and a set of initial conditions. Tobias Mayer (1723-62) used the method to construct his lunar tables, which were an important contribution to the longitude problem.[3]

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[2] MA 290 *Unit 11: Mathematical Physics and the System of the World*, p. 28.
[3] See MA 290 *Unit 11: Mathematical Physics and the System of the World*, p. 23 and MST 121 *Chapter C3: Differential Equations and Modelling*, p. 39.

From the assumption that light always takes the fastest route between any two points, Fermat had used his early calculus to derive Snell's (refraction) law.[4] Later mathematicians saw that the calculus could be used more generally to solve problems in which a quantity is minimised, maximised or conserved. Through the work of Johann and Jacob Bernoulli and Euler, such studies led to the variational calculus. This had interesting practical applications, since in many physical systems energy is conserved.[5] Based on Fermat's ideas, Euler came to think that in all physical systems some quantity or other would be maximised or minimised: "nothing at all takes place in the Universe in which some rule of maximum or minimum does not appear."[6]

An early stimulus in the development of the variational calculus was the brachistochrone problem, posed by Johann Bernoulli in 1696. The problem – solved by Newton, Leibniz, Guillaume de L'Hôpital (1661-1704), Jacob Bernoulli and Johann himself – was to find the curve (the brachistochrone) by which a body descends most quickly from one point to another under the influence of gravity. The curve required was found to be an inverted cycloidal arch.

6. Conclusion

The astonishing range of problems amenable to treatment with the calculus should be clear. But were many (or any) of the practical problems solved in this way the impetus behind the development of the calculus? If by 'practical problems' we mean the sorts of problems that non-mathematicians are (or were) liable to encounter, then it appears the answer must be *no*. Robins and Euler discussed gunnery, but it seems "hard to see these studies as central to the lives of either man."[7] Clairaut's and Euler's work on the moon had practical applications, but their primary motive seems to have been assessment of the Newtonian system and not any

[4] In fact, Leibniz applied the differential calculus to this problem in his first *Acta Eruditorum* article (1684). See **SB** 13.A3, pp. 428-34.
[5] MA 290 *Unit 11: Mathematical Physics and the System of the World*, p. 32.
[6] Leonhard Euler, quoted in Stuart Hollingdale, *Makers of Mathematics* (London: Penguin, 1994), p. 289.
[7] MA 290 *Unit 11: Mathematical Physics and the System of the World*, p. 24.

application to navigational problems. On the other hand, if by 'practical problems' we mean the sorts of problems now tackled in the applied sciences, the answer will be quite different. The development of differential equations and the variational calculus owes much to the physical problems they were used to address.

The unifying theme, however, was that mathematics can help us to understand the workings of the world around us; whether such understanding led to technological innovation was of secondary importance. Furthermore, the initial development of the calculus seems to have been eminently theoretical, concerned only with the geometrical properties of curved lines and surfaces as set out in Section 2, above. This was simply a continuation of the classical geometric tradition. In summary, the development of the calculus from 1660 to 1760 was driven primarily by the desire for increased understanding in three areas: mechanics, geometry, and – especially for Euler – algebraic analysis. It seems the motive to solve the 'practical problems' of non-mathematicians had little to do with it.