A Short Account of Euclid's *Elements*, VI.13[1]

Euclid's *Elements* (c. 300 BC) is probably the most influential mathematical treatise of all time. The *Elements* comprises thirteen books, establishing no fewer than 467 propositions. Euclid's system is *axiomatic*. Each book opens with a statement of *definitions*, *postulates* and *common notions*. Each proposition is established by strict deductive reasoning from these axioms and from other previously established propositions.

Proposition thirteen of book six illustrates these and other general features of the *Elements*. In T.L. Heath's 1925 translation, on which this discussion is based, VI.13 begins with a statement of the proposition: "To two given straight lines, to find a mean proportional". Since book six of the *Elements* "looks at the application of the results of book five,"[2] which itself "expounds the Eudoxan theory of proportion,"[3] VI.13, on constructing a mean *proportion*al, fits neatly into this general scheme.

The task of VI.13 might be presented thus: given two lines AB and BC, find (using only a straightedge and compasses) a third line, such that the length of AB is to the length of that third line as the length of the third line is to the length of BC. While for Euclid this was a geometrical problem, we translate it into an equivalent algebraic problem: Given ***a*** and ***b***, find ***x*** such that …

$$\frac{a}{x} = \frac{x}{b} \;.$$

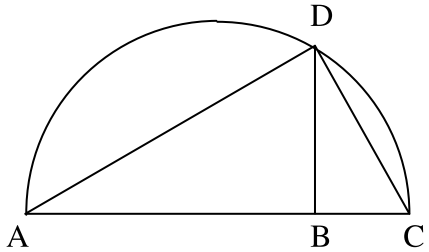
[1] Proposition 13 of Book VI of Euclid's *Elements* can be found in **SB** 3.C4(b), p.124.
[2] J.J. O'Connor and E.F. Robertson, "Euclid of Alexandria," *The MacTutor History of Mathematics Archive* <http://www-history.mcs.st-and.ac.uk/history/Mathematicians/Euclid.html> Accessed 5 March 2003.
[3] Stuart Hollingdale, *Makers of Mathematics* (London: Penguin, 1994), p. 41.

Which problem we solve as follows:

$$\frac{a}{x} = \frac{x}{b}$$
$$x^2 = ab \qquad \{1\}$$
$$x = \pm\sqrt{ab}\ .$$

Euclid placed AB and BC in a straight line, produced a semi-circle on AC, and then constructed a triangle ADC within the semi-circle, as shown in figure 1, below.

<u>*Figure 1*</u>



The points at which the argument depends upon propositions previously established are clearly marked with a back reference to the relevant proposition. By III.31, ADC is a right angle and so, by VI.8, the right-angled triangles ACD, ABD, and BCD are similar. From this it follows that AB is to DB as DB is to BC. Therefore, BD is a mean proportional between AB and BC.[4] To shortcut this last inference Euclid invoked a *porism* or corollary of VI.8, marked as 'VI.8, por.' within the text. Three later propositions (namely VI.25, X.27, and X.28) depend upon VI.13, as does any proposition itself dependent upon any of these three.[5]

The propositions of Euclid's *Elements* fall into two categories: demonstrations and constructions, which end with the letters 'Q.E.D.' and 'Q.E.F.' respectively.[6] Since the problem in VI.13 is to *find* a mean proportional, the proposition ends with 'Q.E.F.'. There is much demonstration even in a construction; Euclid not only shows how to *construct* a mean proportional, he also *demonstrates* that the line so constructed is a mean proportional.

[4] Since Euclid was concerned with geometry, the problem only has a positive solution, where the algebraic version also has a negative solution.
[5] The relations of dependence between the propositions of Euclid's *Elements* can be traced using D. Joyce's online edition at <http://aleph0.clarku.edu/~djoyce/java/elements/Euclid.html>.
[6] Q.E.D. stands for, '*Quod erat demonstrandum*,' Latin for 'which was to be proved'. Q.E.F. stands for '*Quod erat faciendum*,' Latin for 'which was to be done'. See MA290 *Unit 3: The Greek Concept of Proof*, p. 23.

It is interesting to note the equivalence of VI.13 and II.14. The latter tells us how, for any given rectilinear figure, to construct a square with the same area. Due to an earlier proposition (I.45), Euclid only needed to show how to 'square the rectangle'. This amounts to finding a square, of side $\boldsymbol{x}$, equal in area to a rectangle, of sides $\boldsymbol{a}$ and $\boldsymbol{b}$. Since the area of the square is $\boldsymbol{x}^2$, and that of the rectangle $\boldsymbol{ab}$, we are led back to {1}, above. The side of the square is, therefore, a mean proportional between the two sides of the rectangle.

It was probably Hippocrates of Chios (c. 470-10 BC) who reduced the *Delian Problem* of *Doubling the Cube* to that of constructing two mean proportionals.[7] Thereafter, the Delian Problem was always investigated in this form. Hippocrates showed that the cube could be doubled if, given two lines of lengths $\boldsymbol{a}$ and $\boldsymbol{b}$, we could find two lines of lengths $\boldsymbol{x}$ and $\boldsymbol{y}$ such that

$$\frac{a}{x} = \frac{x}{y} = \frac{y}{b} \ .$$

Although the Greeks found several ways to solve this problem, none of these required only a straightedge and compasses. Indeed, in 1837 Pierre Wantzel demonstrated the impossibility of a straightedge and compasses solution.[8]

[7] See the extracts from Proclus (**SB** 2.F2, p 83) and Eutocius (**SB** 2.F3, pp. 83-5).
[8] J.J. O'Connor and E.F. Robertson, "Doubling the Cube" *The MacTutor History of Mathematics Archive* <http://www-history.mcs.st-and.ac.uk/HistTopics/Doubling_the_cube.html> Accessed 23 March 2003.