Pupils’ Misconceptions in Mathematics

> One of the most important findings of mathematics education research carried out in Britain over the last twenty years has been that all pupils constantly ‘invent’ rules to explain the patterns they see around them.
>
> (Askew and Wiliam 1995)

While many of these invented rules are correct, they may only apply in a limited domain. When pupils systematically use incorrect rules, or use correct rules beyond the their proper domain of application, we have a misconception. For example, many pupils learn early on that a short way to multiply by ten is to ‘add a zero’. But what happens to this rule, and to a child’s understanding, when s/he is required multiply fractions and decimals by ten? Askew and Wiliam note that

> It seems that to teach in a way that avoid pupils creating any misconceptions … is not possible, and that we have to accept that pupils will make some generalisations that are not correct and many of these misconceptions will remain hidden unless the teacher makes specific efforts to uncover them.
>
> (1995: 13)

According to Malcolm Swan

> Frequently, a ‘misconception’ is not wrong thinking but is a concept in embryo or a local generalisation that the pupil has made. It may in fact be a natural stage of development. (2001: 154)

Although we can and should steer clear of activities and examples that might encourage them, misconceptions cannot simply be avoided (Swan 2001: 150). Therefore it is important to have strategies for remedying as well as for avoiding misconceptions.

This paper examines a range of significant and common mathematical mistakes made by secondary school children. Descriptions of these mistakes are followed by discussions of the nature and origin of the misconceptions that may explain them. Some strategies for avoiding and for remedying these misconceptions are then suggested. The paper ends by relating some general features of the recommended strategies to the educational theories of Jean Piaget and Lev Vygotsky.

Misconceptions 1 and 2: Algebra
3m + 6 = 9m

The type of error highlighted in the heading for this section arose in each of the three classes to which I have taught algebraic topics. There are at least two misconceptions that could explain this kind of error: (1) that letters represent objects rather than numbers, (2) that an answer should not contain an operator symbol such as +, –, x, or ÷. Both misconceptions are discussed in the following paragraphs.

I encountered the error of simplifying, say, *3m + 6* to *9m* in two different mathematical contexts. These contexts were: (i) expanding brackets containing an unknown and (ii) simplifying expressions by collecting like terms.

Although pupils in one class were only asked to remove the brackets from expressions such as *3(m + 2)*, and not to simplify, some pupils attempted such simplification. The result was that many who correctly expanded *3(m + 2)* to give *3m + 6*, went on to incorrectly simplify this to *9m*.

Being aware that such mistakes often originate in the misconception that numbers represent objects (discussed below), I did everything I could to avoid encouraging such a belief when teaching the collection of like terms. I strongly emphasised the fact that letters represent numbers. In keeping with this, I read *3m* as *three-lots-of-m*; making it clear that *3m* involves multiplying *3* by *m* (or *m* by *3*) and therein implying that the letter represents a number.

The group coped reasonably well until they met examples such as *2m + m + 6*. Many pupils would have responded to this with 9*m* rather than the correct *3m + 6.* Although several pupils were helped by my repetition of such mantras as "adding six is different to adding *six-lots-of-m*," many pupils persisted in this kind of error.

Misconception 1: Letters as Objects

After this lesson the normal class teacher and a second observer suggested that perhaps I would have had more success with a different 'more concrete' approach. Both recommended that in future I use what others have called the 'fruit salad' approach (Tirosh et al. 1998).

This approach involves reading *3a + 2b + b + 4a* as *3 apples plus 2 bananas plus a banana plus 4 apples*, which naturally becomes 7 apples plus 3 bananas or *7a + 3b*, which is correct. Unfortunately this would have helped least where I needed help most, and it would also have encouraged the same mistakes!

As Doug French points out, if *k* is interpreted as kangaroos, when faced with the expression $2k + k + 4$, the students' "instinctive thought is likely to be '4 what?'. And the obvious answer is 4 kangaroos, giving $7k$ altogether" (2002: 11).

This is not the only problem that can arise from thinking of letters as representing objects. If *a* and *b* represent apples and bananas, then what does *ab* mean? Indeed, if letters are used in this manner and we are asked to convert the equivalence *a week is seven days* into a mathematical formula relating days and weeks, we could easily be led to $w = 7d$ rather than $d = 7w$ (where *w* is the number of weeks, and *d* the number of days, in a given period).

Given the popularity of the fruit salad approach among teachers at the school in question, I would not be surprised if such a misconception was the underlying cause of much of the confusion on the topic of expanding brackets. However, my experience with the collection of like terms led me to search for other possible explanations for such errors.

Misconception 2: Operations vs Answers

My conclusion, confirmed by the literature (Tirosh et al. 1998), was that a large part of the problem lay in the fact that to many pupils $3m + 6$ does not look like an answer. The presence of the operator symbol, +, makes the 'answer' appear unfinished. In short, pupils see such symbols as +, –, x, and ÷ as invitations to *do something*, and if something is still to be done, then they ought to do it. If we have to remove the symbol by doing what it tells us to do, in this case adding, it is only natural that $3m + 6$ should become $9m$.

Readers who find it difficult to understand this tendency may like to consider their own response to the following statement

$$3 \div 40 = {}^{3}/_{40}$$

Most people respond differently to the two sides of this equation. The left-hand-side looks like a question, namely "What is three divided by forty?" The right-hand-side, however, is not a question but simply a fraction: three fortieths. In a sense these are just two different ways of writing the same thing, but this may also be seen as revealing that in both arithmetic and algebra some expressions lead a dual existence as both process and product. $3m + 6$ can be seen as a set of instructions for calculating a numerical value, but also as mathematical object in its own right (French 2003).

The resistance to accepting *3m + 6* as an answer is easily understood. In ordinary arithmetic it is always possible to remove the operator signs (unless there are infinite in number), and the final answer has not been reached until they are all gone..

Strategies and Remedies

For reasons that should already be clear, despite its obvious appeal the 'fruit-salad' approach seems best avoided. It may also be possible to over use 'realistic' contexts in the early stages of teaching algebra, as such contexts often lead us (and textbook authors) to use letters in ways that might invite a letters-as-objects interpretation (*m* means *miles*, *h* means *height* and the like). In the place of such 'realistic' examples, I would suggest using number puzzles, tricks and games. Pupils find these very engaging and algebra can be a very powerful tool for solving and explaining in these contexts.

In my more recent teaching of collecting like terms, I began the topic with a 'think of a number' trick, requiring a sequence of mental operations. When the results were collected in, nearly everyone got the same answer. Those who did not were quickly helped to realise their mistakes. Algebra was then introduced as a tool that could explain why this happened, why it is that we could all start with different numbers, but all reach the same final answer after performing the operations in question. The topic of collecting like terms fits neatly with such tricks, as the simplifications work 'whatever numbers the letters represent', just as in the trick everyone reaches the same answer 'whatever number they start with'.

After working through some examples on the board, including some of the problematic form *2m + m + 6*, I stopped to ask the class what is 'funny' about the answer *3m + 6*. The class immediately volunteered that it is funny because it contains a plus symbol. After commending this response, I briefly commented that "this happens sometimes in algebra", and set the class to work through some questions for themselves. When one student became stuck on a question of this form, all I said was "It's another of those ones with a funny answer" and he immediately knew how to continue.

Another useful approach with students who have already made such errors is to substitute a particular value (or values) into the two expressions believed to be equivalent (French 2002: 12). This approach has the benefit of bringing about a “cognitive conflict” (Swan 2001). On the one hand the pupil believes that the two expressions are equivalent, on the other they can see that they give different results when values are substituted into them. In this situation the pupil can see that something has to be wrong, and can even be invited to explain and resolve this conflict for him- or herself. Even if the pupil is unable to resolve this conflict, the awareness of it is likely to make him or her more receptive to resolutions offered by the teacher or by other students.

Substitution seems especially powerful in communicating equivalence (and the lack thereof) when each of a set of values is substituted into the expressions in question. Two expressions are equivalent if, and only if, they give the same result for all values that we choose to substitute.

Misconceptions 3 and 4: The Equals Sign
Equals as Makes / Equals as a Logical Connective

The first misconception to be discussed in this section has been present in almost every class I have taught or observed, including high and low attaining groups in all years. One instance occurred in a Y10 lesson on finding fractions of a quantity. Students were taught a standard method for tackling such problems as “Find ¾ of 16” The method is to first find ¼ of 16 (by dividing 16 by 4) and then multiply the result by 3. In solving this problem, several pupils in the class would have written the following: $16 \div 4 = 4 \times 3 = 12$. While the answer reached is correct, the written statement is not. That statement (unlike the pupil) asserts, among other things, that $16 \div 4 = 4 \times 3$, which is equivalent to asserting that $4 = 12$, which is clearly false. The same problem occurs with any two-step calculation. A similar but importantly different error is often seen in algebraic contexts, Doug French points out that “it is common to see erroneous statements like: $3x - 5 = 7 = 3x = 12 = x = 4$” (2002: 14). Again, this statement asserts both that $12 = x$ and that $x = 4$, and therefore entails the falsehood that $12 = 4$. Both of these mistakes can be explained in terms of alternative interpretations of the equals sign.

Misconception 3: Equals as Makes

The misconception behind the first kind of mistake is well summarised by Candia Morgan, who observes that there is a

> persistent use of an ‘operator’ concept of the equals sign by students at all levels which suggests that it is fulfilling a material rather than a relational role.
>
> (1998: 81)

In writing 16 ÷ 4 = 4 x 3 = 12 pupils seem to show that they use the equals sign not to state an equivalence between two expressions but as an instruction to calculate the value of the previous expression. So, in our example it is as though the two equals signs connect only the emboldened portions in which they are contained

16 ÷ 4 = 4 x 3 = 12.
16 ÷ 4 = **4 x 3 = 12**.

In each of these cases the term following the equals sign does indeed give the value of the previous expression. Doug French explains the misconception and its likely origin as follows:

> Students interpret equals as an instruction to do something to determine a result rather than as a symbol that indicates the equivalence of two expressions. This arises in a natural way through the use of equals in numerical calculations. It is also encouraged by the presence of a key labelled with an equals sign on many calculators. (2002: 13-4)

Misconception 4: Equals as a Logical Connective

The mistake made in the algebraic context is different, and involves the equals sign playing a dual role. In some contexts it functions as we would expect, but in others it plays the role of such words as *therefore*, *so*, *leads to*, or *entails*. Using French’s example, we can easily restructure the incorrect statement into a well formed argument by replacing every other equals sign with a genuine connective. In this way, $3x - 5 = 7 = 3x = 12 = x = 4,$ becomes

$$\begin{array}{lrcl} & 3x-5 & = & 7 \\ \textit{So,} & 3x & = & 12 \\ \textit{Hence,} & x & = & 4. \end{array}$$

The above quoted passage from Morgan continues:

> Another common role [for the equals sign] is as a logical connective between statements, for example 5x + 3 = 2x − 15 = 5x = 2x − 18. [This is] likely to be considered to be mathematically incorrect by a secondary school teacher-assessor; a recent handbook for mathematics teachers … picks out use of the equals sign as a connective as its single “example of bad practice” … to illustrate writing that does not ‘make sense when read aloud’ (1998: 81)

Strategies and Remedies

The comment at the end of the above quote suggests that one possible remedy for such mistakes is the practice of reading the offending statement aloud. I would suggest that when using this strategy, it might sometimes be valuable to replace the word *equals* with a synonym such as *is-the-same-as*.

This strategy is intended to help the pupil see the logical implications of their claim. In the case of $16 \div 4 = 4 \times 3 = 12$, it might help the pupil to see that their written statement implies that $16 \div 4 = 12$, which they know is false. Again, this has the benefit of creating a cognitive conflict, a conflict between their mental methods and the implications of the accompanying written methods. This conflict can then be used to encourage the accepted interpretation and use of the equals sign.

It is also important to model the correct use of the equals sign in your board-work. If the teacher uses the sign incorrectly, this can only increase the likelihood of pupils doing the same. Other useful strategies include emphasising the symmetry of the equals sign, for example by not reversing the final line in the following piece of working out:

$$\begin{aligned} 2x &= 3x - 3 \\ 2x + 3 &= 3x \\ 3 &= x \end{aligned}$$

It may also be useful to limit pupils to one equals sign per line.

Misconception 5: The Addition of Fractions
Add the tops, add the bottoms

In this section I consider an error that I encountered in teaching the addition (and subtraction) of fractions to both Y8 and Y9. The error in question is to 'add' two or more fractions by simply adding the numerators and adding the denominators as follows.

$$\frac{3}{4} + \frac{2}{3} = \frac{3+2}{4+3} = \frac{5}{7}$$

With both classes, I built towards teaching the correct method by first considering examples where the fractions were already written on a common denominator, and after a lesson on equivalent fractions, moved on to consider cases such as the above. Despite their success in the first two lessons, many pupils found the third lesson extremely difficult, in most cases because they struggled to find appropriate common denominators. During the lesson, then, nearly all pupils attempted to follow the correct method, although they found it difficult to use. Later homeworks and tests revealed a tendency to use the erroneous ‘add the tops, add the bottoms’ technique. My overall judgements about these classes were very much in line with the conclusions of K. Hart.

> A very common error in the addition of fractions was to use a rule ‘add tops add bottoms’. This … was more prevalent in examples where the two denominators were different. It was also interesting to note that this particular error occurred more when the question was posed in computation form than in [word] problem form. (Hart 1981a: 75)

Misconception 5: Add the tops, add the bottoms

In older children, who have met the multiplication of fractions, this error might be explained as a ‘local generalisation’ that involves assuming that we apply any given operation to a pair of fractions simply by applying that operation to the numerators and denominators taken separately. Unfortunately, the pupils in question had not yet met the multiplication of fractions, and so the explanation is false in this case. It may, however, be true in others. Indeed, this might explain one of Hart’s other observations: “The ability to solve addition and subtraction computations [with fractions] *declines* as the child gets older” (1981a: 79).

A more likely explanation for pupils adopting this procedure is that they conceive fractions as test-scores, at least when written in their standard notation. Indeed, a test score of 32 out of 50 is usually written in the form $^{32}/_{50}$. As Malcolm Swan (2001: 149) points out, when we add two test scores together we rightly adopt the rule of adding the tops and bottoms; a score of $^{10}/_{20}$ on one test and $^{15}/_{20}$ on another *does give* an overall score of $^{25}/_{40}$.

Strategies and Remedies

As with misconceptions 3 and 4, a useful strategy in this case is to encourage pupils to read the question aloud, and to read it in a way that naturally suggests the correct approach. The addition

$$\frac{1}{5} + \frac{3}{5}$$

should not be read as *one-over-five plus three-over-five*, but rather as *one-fifth plus three-fifths*. Fifths are objects that most pupils are comfortable adding; one of them plus three of them is obviously four of them in total.

A cognitive conflict strategy is also available for this misconception. The vast majority of pupils can tell you that a half plus a half is one (or one whole). But if you get the pupils to write down the question and apply their general strategy, something interesting happens:

$$\frac{1}{2} + \frac{1}{2} = \frac{1+1}{2+2} = \frac{2}{4} = \frac{1}{2}\ .$$

I used this strategy with some success with Y8. This example might not be sufficient to shake the misconception, as it involves adding two fractions with the *same* denominator. In that case we might also try adding a half to three-quarters. Pupils will be able to see that the answer should be more than one, and that their method gives an answer of $^5/_6$, which is less than one. Again, the experience of cognitive conflict will not only lead the pupil to seek a resolution of their own but will make them more receptive the resolutions proposed by others.

Misconception 6: Reflection
The same, but on the other side of the line

In an end of term test, I recall watching one pupil as she attempted to answer the following question in Figure 1 (all figures are at the end of this document). The girl's initial answer is shown in figure 2. I was pleased to see her rub this answer out, believing that she was about to replace this answer with a correct one. Unfortunately she only rubbed it out to increase the length of the rectangle she had drawn. In fact, she did this several times before she was happy that she had done it correctly. Many other pupils in the class made the same error. Pupils performed much better on the other two reflection questions, in which the mirror-lines were vertically and horizontally orientated.

Misconception 6: The same, but on the other side of the line

Dietmar Küchemann confirms that mistakes of this kind are common: "Children may have particular difficulties when the mirror-line is not horizontal or vertical but the slope of the object is horizontal/vertical" (Küchemann 1981: 143). The APU (1980)) studies revealed a similar tendency when the mirror-line is horizontal or vertical but the slope of the object is not. For example, many respondants answered the question in Figure 3 as shown in Figure 4.

The examples given here suggest that pupils may understand reflections as 'completing' a shape, by making it the same on both sides of the mirror-line. However, examples discussed by Küchemann reveal that the given shape does not need to meet the mirror line for this kind of mistake to occur.

Reflecting shapes in vertical and horizontal lines is certainly easier than reflecting them in lines of any other orientation. Reflections are also easier when the slope of the object is oriented either parallel or perpendicular to the mirror-line. Understandably, teaching on reflection often begins with examples of this sort. In a context such as this, many of the resulting reflections can be explained by a rule that does not extend to more complex cases. The rule is this: flip the object over in such a way as to leave it's slope unchanged, now translate the object in the direction of its slope until it lies on the other side of the mirror-line, and at the same distance from that line. This is much easier to do than to explain. This rule fits the case of Figures 3 and 4 exactly, and makes the answer in figure 2 appear quite natural … the slight lack of fit combined with the obvious fact that the object (and so the image) meets the mirror-line in two places, also seems to explain the girl's dithering over exactly what length to make the resulting rectangle.

Strategies and Remedies

Küchemann suggests that these misconceptions may be best avoided by tying the idea of reflection to that of folding, and the essential properties of reflection teased out through investigative work.

> The actions and the representations are both highly intuitiable so that it should be possible to develop such an approach in ways that are meaningful to most children. The transformations can be internalised in gradual steps, by focussing first on the actions themselves … and then on the representation of imagined actions. In addition, the resulting drawings can be checked at each step by a return to the actions. … The approach advocated is one that directs children towards discoveries from which the rules and properties of the transformations can be surmised and against which they can be tested. (Küchemann 1981: 157)

A similarly investigate approach could be used in conjunction with a dynamic geometry package. This would allow students to investigate what properties of a "reflection" remain unchanged as the image, object and mirror-line are manipulated in various ways. The value of such investigative, discovery based approaches will be discussed in the next section.

Misconceptions and Theories of Learning

Many of the above strategies and remedies fit well with the theories of learning due to Jean Piaget and Lev Vygotsky. Due to his emphasis on personal interactions, particularly in the use of language, Lev Vygotsky's theory of learning is often thought of as a sociocultural theory. Vygotsky believed that the ideal learning relationship was that of a novice learning as an apprentice under the tutelage of an expert. Nevertheless, like Piaget, he would have seen much value in peer discussions that, under the guidance of a teacher, overcome differences of opinion and in doing so achieve a shared understanding.

Vygotsky's theories also provide a theoretical backing for strategies involving reading aloud. What is at first read aloud is, of course, soon expected to become self-speech (also referred to as private-speech or inner-speech). According to Vygotsky, as the principle form of self-direction and self-regulation, such self-speech is the foundation of all higher cognitive powers (Berk 1997: 248).

According to Piaget, all cognitive change can be classified as one of two types: *adaptation* and *organisation*. *Organisation* is a largely internal process involving rearranging and linking up items of previous learning to form a "strongly interconnected cognitive system" (Berk 1997: 213). More important for our purposes is *adapation*, which itself comes in two varieties: *assimilation* and *accommodation*. In assimilation the learner simply fits new concepts, skills and information into his or her existing cognitive framework. However, on some occasions new items of learning cannot be fitted into the existing cognitive framework, and that framework must be changed in order to make room for them. This is *accommodation.*

The awareness of a need for a change in one's cognitive framework is brought about by a realisation that something important 'doesn't fit in'. For this reason, Malcolm Swan and others in the Diagnostic Teaching Project have seen Piaget's views as providing theoretical justification for their view that the best way to overcome a misconception is by engineering a cognitive conflict (Swan 2001).

Many of the strategies suggested above fit into this category. When those strategies were described, they were described as strategies for use with individual students. However, Swan and others suggest that the strategy may be as, if not more, effective when the conflict in question is between members of the same peer group, who must then come to a resolution through discussion.

> A [pupil] might superficially accept a [teacher's] perspective without critically examining it, out of an unquestioning belief in the [teacher's] authority. Piaget also asserted that clashing viewpoints – arguments jarring the [learner] into noticing a peer's point of view – were necessary for peer interaction to stimulate movement towards logical thought. (Berk 1997: 253)

Fortunately, many of the strategies suggested earlier could also be used to stimulate debate in a group or whole-class teaching situation. Interestingly, these same general techniques are much less successful when used as preventative strategies:

> Addressing misconceptions during teaching does actually improve achievement and long-term retention of mathematical skills and concepts. Drawing attention to a misconception before giving the examples was less effective than letting the pupils fall into the 'trap' and then having the discussion.
>
> (Askew and Wiliam 1995: 13)

This element of peer discussion is also related to Piaget's emphasis of discovery learning (as opposed to straight forward teacher instruction). According to Piaget, we learn better those things that we discover for ourselves. Discovery enables a person to own their beliefs, ideas and concepts in a way that transmission models of teaching do not. In this regard Küchemann's strategies for teaching the topic of reflection seem ideal, and indeed provide a paradigm example of effective teaching methods.

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Figure 1

Copy this diagram onto squared paper and draw the reflection in the mirror shown by the dotty line

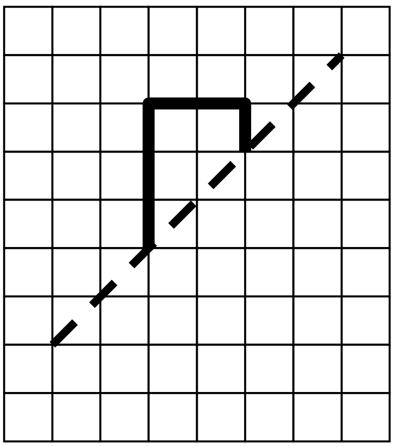

Figure 2

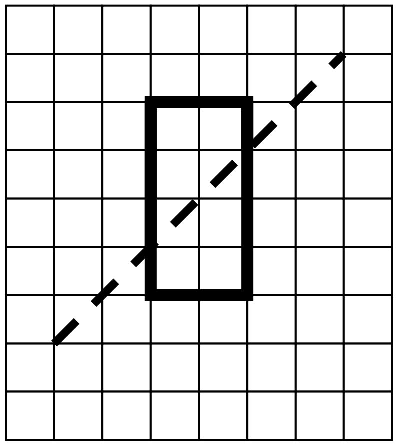

Figure 3

Reflect the blue line in the dotted mirror line

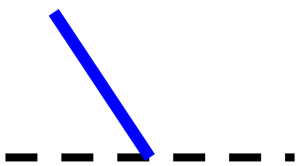

Figure 4